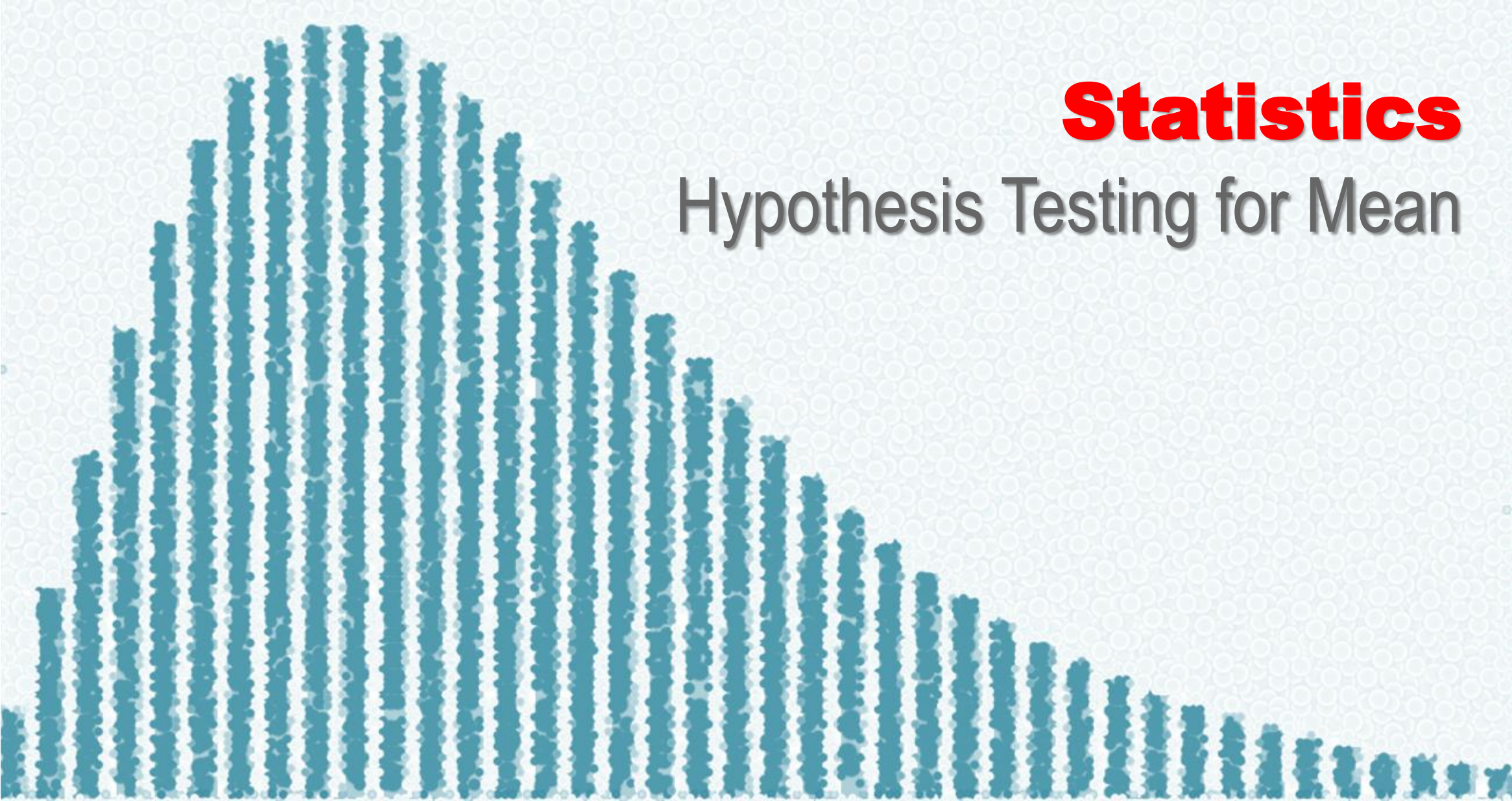




Statistics

Hypothesis Testing for Mean

Contents

Part 1 Population Parameter vs. Sample Statistic

Part 2 The Normal Distribution

Part 3 Hypothesis Testing for Mean

Part 4 Confidence Interval

Learning Objectives

By the end of this chapter, students should be able to

1. distinguish between population parameters and sample statistics,
2. present a summary of sample data using a computer software,
3. describe the properties of a normal distribution,
4. determine the confidence interval of a population mean,
5. perform hypothesis tests for a claim made about a population mean.

Introduction

What is Statistics?

Why learn Statistics?

Who uses Statistics?

What is Population? Sample?



Part 1. Population Parameter vs. Sample Statistic

Population refers to ALL data we are concerned with. **Parameters** are measurements taken from population data, e.g. population mean (μ) and population standard deviation (σ).

Sample is a subset of the population. **Statistics** are measurements taken from sample data, e.g. sample mean ($\bar{x}$), sample standard deviation (s) and sample size (n).

POPULATION

mean = μ (mu)

standard deviation = σ (sigma)

SAMPLE

mean = $\bar{x}$ (x bar)

standard deviation = s

Part 1. Population Parameter vs. Sample Statistic

Measure of Centre and Measure of Spread

Population mean

VS

Sample mean

$$\mu = \frac{x_1 + x_2 + \cdots + x_N}{N}$$

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Population standard deviation

VS

Sample standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^N (x_i - \mu)^2}{N}}$$

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}}$$

Part 1. Population Parameter vs. Sample Statistic

Consider two sets of sample data.

Set A: 24 25 26 27 28
Set B: 5 17 20 30 58

	A	B
1	Set A	Set B
2	24	5
3	25	17
4	26	20
5	27	30
6	28	58
Sample Mean, $\bar{x}$	26	26
Sample Std. Dev., s	1.581	19.987

Notice this (including the Excel formulas):

1. Both sets of data give the same sample mean
2. However, Set B gives a larger standard deviation (= wider data spread)
3. Conversely, Set A gives a smaller standard deviation (= narrower data spread)

=AVERAGE(A2:A6) for Set A, =AVERAGE(B2:B6) for Set B

=STDEV.S(A2:A6) for Set A, =STDEV.S(B2:B6) for Set B

Part 2. The Normal Distribution

The *probability* of an event A happening is given by:

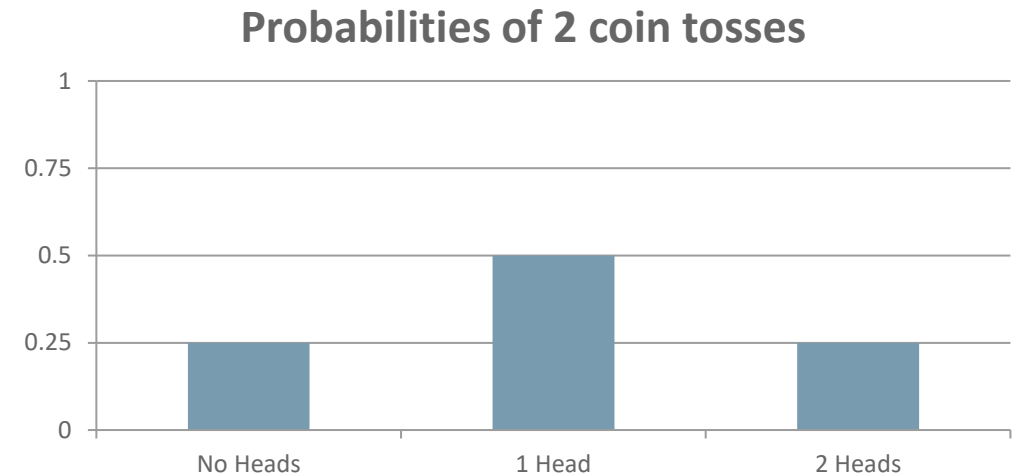
$$P(A) = \frac{\text{Number of ways that } A \text{ can occur}}{\text{Total number of possible outcomes}}$$

- $0 \leq P(A) \leq 1$ for any event A .
- $P(A) = 0 \Rightarrow$ The event A is **impossible** to occur.
- $P(A) = 1 \Rightarrow$ The event A **will certainly** occur.

Part 2. The Normal Distribution

Probability distribution is a description of the probabilities of all possible events.

Consider the number of heads obtained from two coin tosses. There are three possible events: 0, 1, 2. Such data that takes on countable outcomes is called *discrete*. The probability distribution of a discrete data can be visualized by using bar chart.

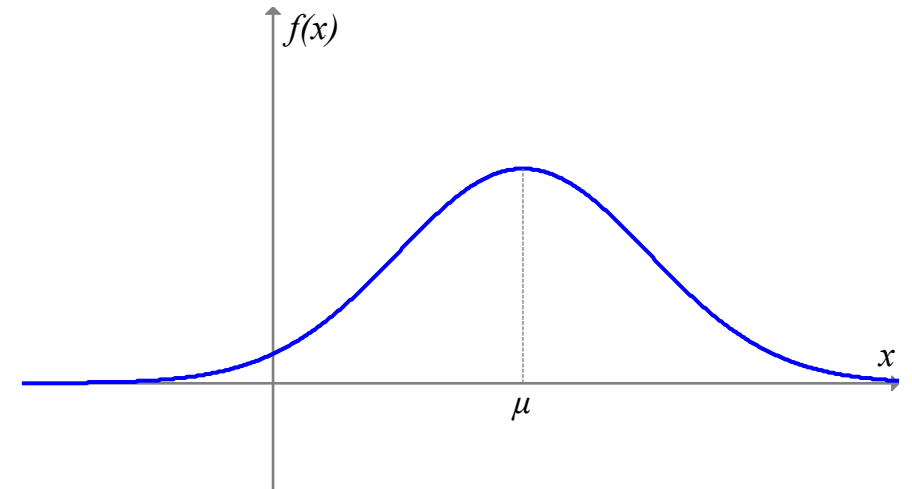


Part 2. The Normal Distribution

Probability distribution is a description of the probabilities of all possible events.

Consider another scenario where the hourly temperature at a certain place is recorded. Such data that takes on any number within an interval is called *continuous*.

The probability distribution of a continuous data is called *probability density function (PDF)*.



PDF of the hourly-recorded temperature

The PDF graph above is a bell-shaped curve called the *normal curve*.

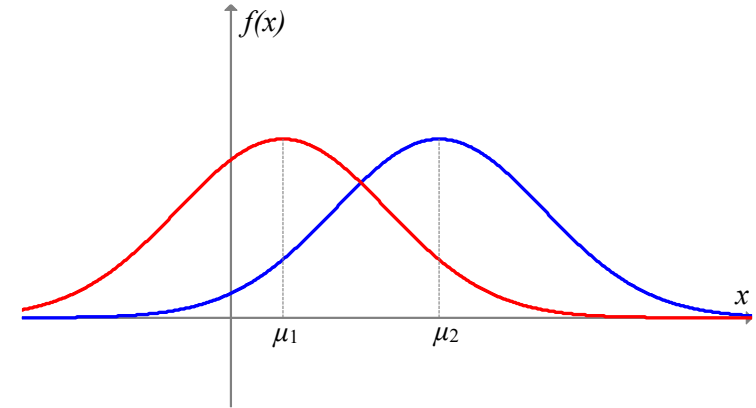
Part 2. The Normal Distribution

A normal curve has the following properties:

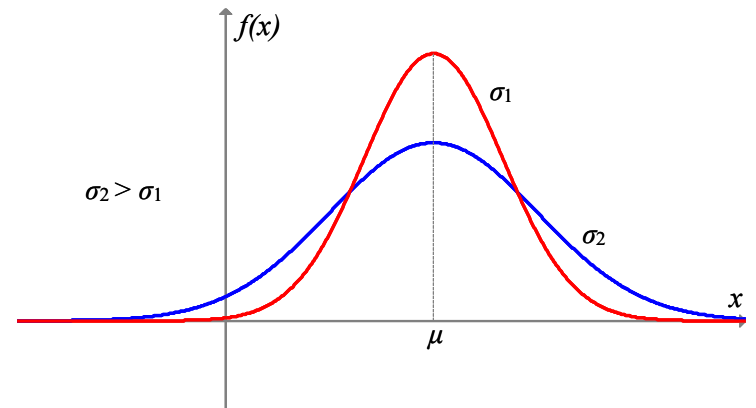
1. It is symmetric and reaches its peak at $x = \mu$ then tapered-off towards zero.
2. The probability $P(a < x < b)$ is the area under the curve of the PDF $f(x)$ over the interval $a < x < b$, i.e.

$$P(a < x < b) = \int_a^b f(x) dx$$

Integral of f from a to b , will be discussed in the later chapter of Integration

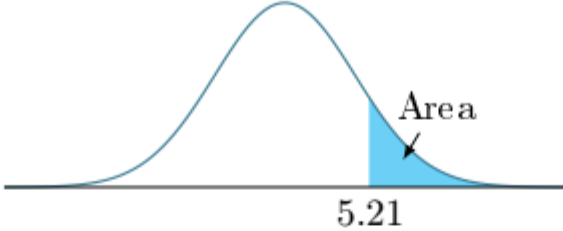
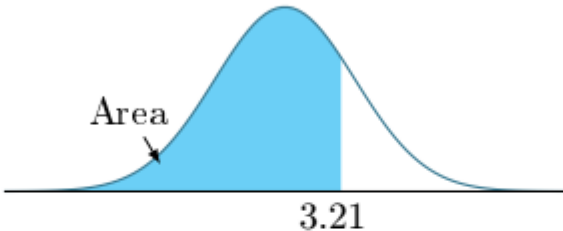
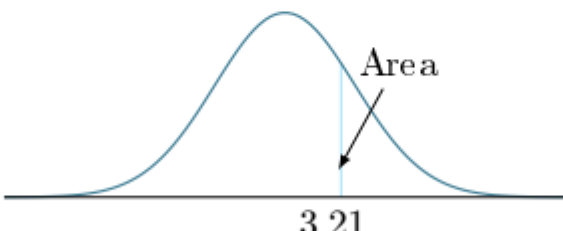
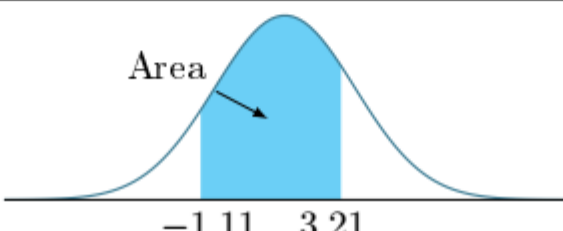


Two normal PDFs with different μ but same σ



Two normal PDFs with same μ but different σ

Part 2. The Normal Distribution – Some examples

$\mathbb{P}(X > 5.21)$		This is the area to the right of 5.21.
$\mathbb{P}(X < 3.21)$		This is the area to the left of 3.21
$\mathbb{P}(X = 3.21)$		The area is zero.
$\mathbb{P}(-1.11 < X < 3.21)$		$\mathbb{P}(X < 3.21) - \mathbb{P}(X < -1.11)$

Part 2. The Normal Distribution

The normal curve also known as the bell curve or the *Gaussian curve* after the German mathematician Carl Friedrich Gauss (1777 – 1855) and its probability distribution is an example of a normal distribution.

THE NORMAL DISTRIBUTION

Let X be a continuous variable with mean μ and standard deviation σ . Then X has a **normal distribution** if X has the normal PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty.$$

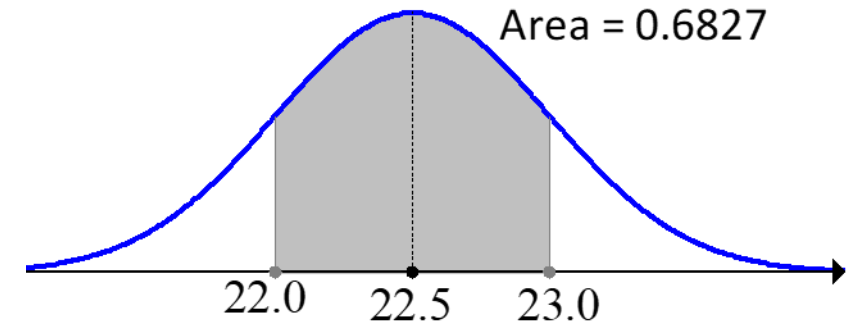
The **standard normal distribution** is a normal distribution with $\mu = 0$ and $\sigma = 1$. Any normal distribution X can be made into standard normal distribution Z through the transformation $Z = \frac{X-\mu}{\sigma}$.

Part 2. The Normal Distribution

EXAMPLE 2.1

X is distributed normally with a mean of 22.5 and a standard deviation of 0.5. Calculate the probability $P(22.0 < X < 23.0)$.

	A	B
1	$\mu =$	22.5
2	$\sigma =$	0.5
3	$X_{\text{low}} =$	22.0
4	$X_{\text{high}} =$	23.0
5	$P(X < X_{\text{low}}) =$	0.1586=NORM.DIST(B3,B\$1,B\$2,TRUE)
6	$P(X < X_{\text{high}}) =$	0.8413=NORM.DIST(B4,B\$1,B\$2,TRUE)
7	$P(X_{\text{low}} < X < X_{\text{high}}) =$	0.6827=B6-B5



Part 2. The Normal Distribution

EXAMPLE 2.1

X is distributed normally with a mean of 22.5 and a standard deviation of 0.5.
Calculate the probability $P(22.0 < X < 23.0)$.

	A	B	C	D
1	$\mu =$	22.5	$Z = (X - \mu) / \sigma$	
2	$\sigma =$	0.5		
3	$X_{\text{low}} =$	22.0	$Z_{\text{low}} =$	-1
4	$X_{\text{high}} =$	23.0	$Z_{\text{high}} =$	1
5	$P(X < X_{\text{low}}) =$	0.1586	$P(Z < Z_{\text{low}}) =$	0.1586=NORM.S.DIST(D3)
6	$P(X < X_{\text{high}}) =$	0.8413	$P(Z < Z_{\text{high}}) =$	0.8413=NORM.S.DIST(D4)
7	$P(X_{\text{low}} < X < X_{\text{high}}) =$	0.6827	$P(Z_{\text{low}} < Z < Z_{\text{high}}) =$	0.6827

Using the transformation

$$Z = \frac{X - \mu}{\sigma}$$

Part 3. Hypothesis Testing for Mean

Hypothesis testing for mean is a statistical procedure that uses sample data to test a claim about population mean.

Null hypothesis

A statement that the value of the population mean μ is **equal to** some claimed value.

$$H_0: \mu = \mu_0$$

vs

Alternative hypothesis

A statement that the value of the population mean μ **differs from** the value stated in the null hypothesis.

$$H_1: \mu \neq \mu_0, \quad \text{or}$$

$$H_1: \mu < \mu_0, \quad \text{or}$$

$$H_1: \mu > \mu_0$$

Part 3. Hypothesis Testing for Mean – In greater explanation

Null hypothesis

$$H_0: \mu = \mu_0$$

Symbol: Typically denoted as H_0 .

The null hypothesis (H_0) is a statement of no effect or no difference in a population parameter.

Purpose: It serves as a baseline assumption that there is no relationship, effect, or difference between groups or conditions being compared.

vs

Alternative hypothesis

Either $H_1: \mu \neq \mu_0$, or, $H_1: \mu < \mu_0$, or, $H_1: \mu > \mu_0$

Symbols: Denoted as H_1 or H_a , and it can be

one-tailed	two-tailed
Indicating a direction of the effect. $H_1: \mu > \mu_0$ $H_1: \mu < \mu_0$	Indicating a difference but not specifying the direction, e.g. $H_1: \mu \neq \mu_0$

The alternative hypothesis contradicts the null hypothesis, proposing that there is a relationship, effect, or difference in a population parameter.

Purpose: It represents the researcher's hypothesis or the effect they are trying to detect in their study.

Part 3. Hypothesis Testing for Mean – In greater explanation

Question: Do post-secondary students in Singapore receive an average of \$14 pocket money per day?

Question inspired by *The Straits Times* article [How much pocket money do Singapore children get? It's \\$4 to \\$14 a day.](#) (Published 26 May 2024)

Null hypothesis ($H_0: \mu = \mu_0$)

The average/mean (*choose either word*) daily pocket money received by post-secondary students in Singapore **is \$14**

$$H_0: \mu = 14$$

Alternative hypothesis (depends on whether we are working with **either** two-tailed **or** one-tailed test [left **or** right-tailed])

Two-tailed ($H_1: \mu \neq \mu_0$)

The avg/mean daily pocket money received by post-secondary students in Singapore **is not \$14.**

$$H_1: \mu \neq 14$$

or

One-Tailed ($H_1: \mu < \mu_0$ or $H_1: \mu > \mu_0$)

The avg/mean daily pocket money received by post-secondary students in Singapore **is less than \$14.**

$$H_1: \mu < 14 \text{ (Left-tailed test)}$$

or

The avg/mean daily pocket money received by post-secondary students in Singapore **is more than \$14.**

$$H_1: \mu > 14 \text{ (Right-tailed test)}$$



Part 3. Hypothesis Testing for Mean

Sample Statistics

Sample statistics are numerical values calculated from the sample data and are used to describe the sample.

Examples:

$\bar{x}$ = sample mean, s = sample standard deviation, n = sample size

Used to calculate

Test Statistic

The test statistic is calculated using the relevant sample statistics, and is then subsequently used to determine p-value.

Test statistic: $z_0 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ or $t_0 = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

A value that indicates how far the sample mean is from the population mean stated in the null hypothesis.

Part 3. Hypothesis Testing for Mean – Significance Level & P-value

Recall: Hypothesis testing for mean is a statistical procedure that uses sample data to test a claim about population mean.

Significance Level

A small positive value, denoted by α (alpha), where $0 < \alpha < 1$, chosen before carrying out the hypothesis test. Used as a threshold to determine whether the evidence against the null hypothesis is statistically significant.

Common values for α : $\alpha = 0.01$ or $\alpha = 0.05$ or $\alpha = 0.1$

Hint for students: Usually given to us! 😊 (What is the α value for our EM2 assignment?)

P-value

A probability value, denoted by p , where $0 < p < 1$ calculated using the test statistic obtained from the sample data. It is used to measure the strength of evidence against the null hypothesis.

$p\text{-value} \leq \alpha \Rightarrow$ Sufficient evidence to reject H_0
 $p\text{-value} > \alpha \Rightarrow$ Insufficient evidence to reject H_0

Part 3. Hypothesis Testing for Mean – P-value (Two-tailed, $H_1: \mu \neq \mu_0$)

Recall: Hypothesis testing for mean is a statistical procedure that uses sample data to test a claim about population mean.

P-value
(Two-tailed)

For a **two-tailed test**, the p-value accounts for extreme results in both tails of the distribution:

$$2P(z > |z_0|)$$

or

$$2P(t > |t_0|)$$

We arrive at either of the following conclusions:

- Sufficient evidence to reject the null hypothesis if the p-value $\leq \alpha$, or,
- Insufficient evidence to reject the null hypothesis if the p-value $> \alpha$.

Part 3. Hypothesis Testing for Mean – P-value (One-tailed test with $H_1: \mu < \mu_0$)

Recall: Hypothesis testing for mean is a statistical procedure that uses sample data to test a claim about population mean.

P-value
(One-tailed)

For a **left-tailed test**, the p-value accounts for extreme results in the left tail of the distribution:

$$P(z < z_0)$$

or

$$P(t < t_0).$$

We arrive at either of the following conclusions:

- Sufficient evidence to reject the null hypothesis if the p-value $\leq \alpha$, or,
- Insufficient evidence to reject the null hypothesis if the p-value $> \alpha$.

Part 3. Hypothesis Testing for Mean – P-value (One-tailed test with $H_1: \mu > \mu_0$)

Recall: Hypothesis testing for mean is a statistical procedure that uses sample data to test a claim about population mean.

P-value
(One-tailed)

For a **right-tailed test**, the p-value accounts for extreme results in the right tail of the distribution:

$$P(z > z_0)$$

or

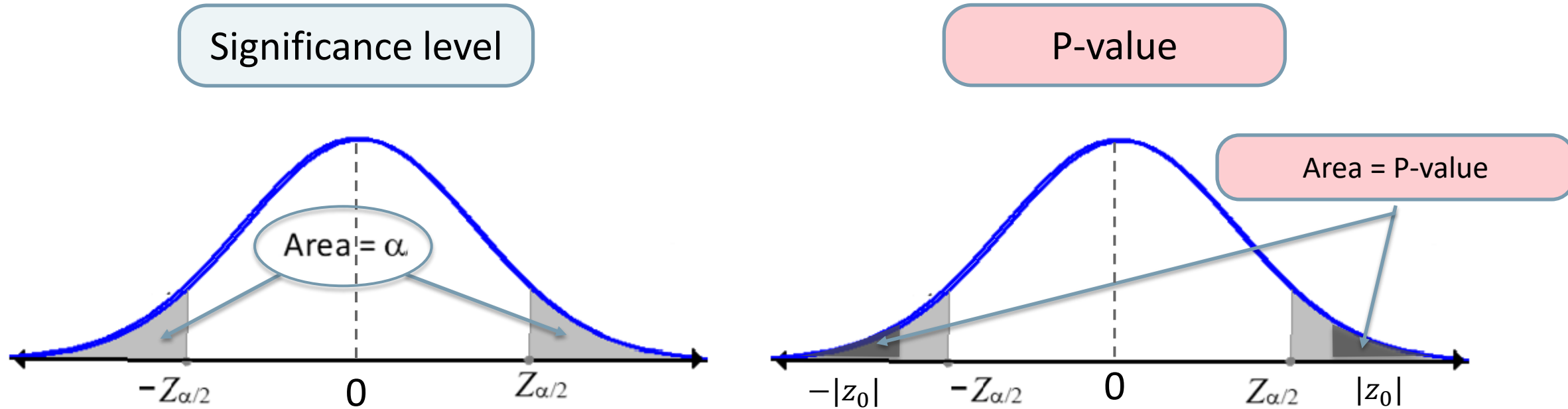
$$P(t > t_0).$$

We arrive at either of the following conclusions:

- Sufficient evidence to reject the null hypothesis if the p-value $\leq \alpha$, or,
- Insufficient evidence to reject the null hypothesis if the p-value $> \alpha$.

Part 3. Hypothesis Testing for Mean (Two-tailed, $H_1: \mu \neq \mu_0$)

Graphical illustration of significance level vs. p-value for a two-tailed test:

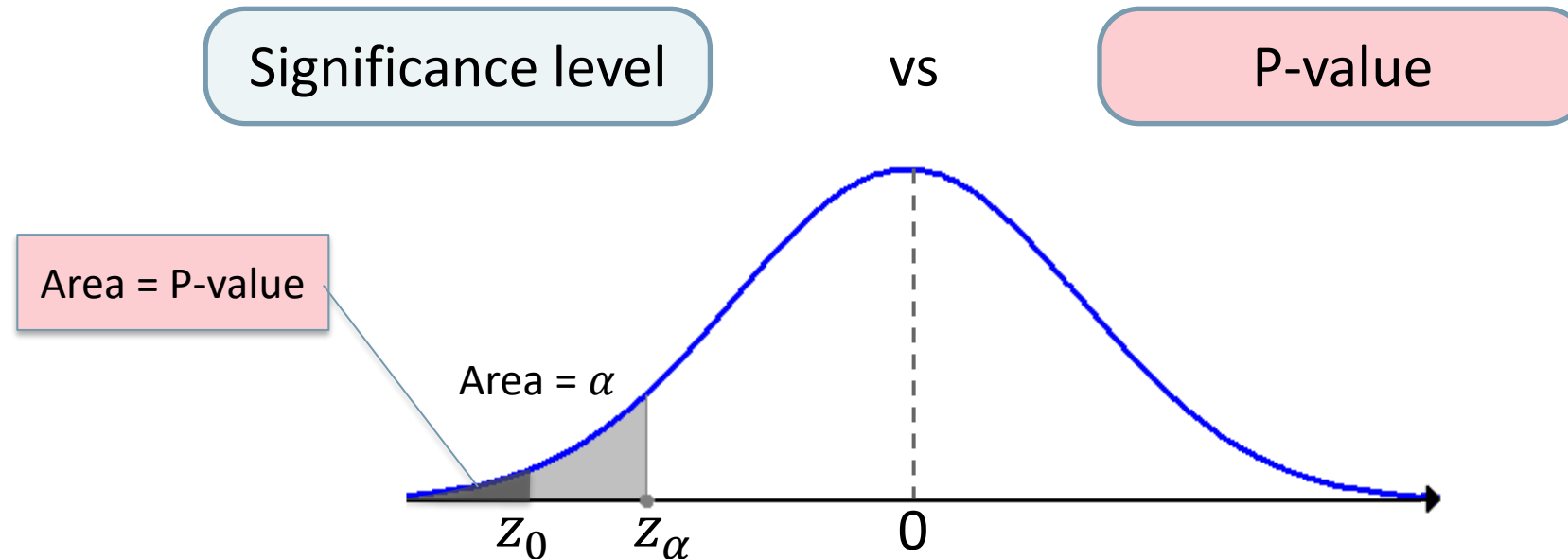


When the p-value $\leq \alpha$, the observed test statistic is sufficiently far from 0, which provides sufficient evidence to reject the null hypothesis H_0 .

Conversely, when the p-value $> \alpha$, there is insufficient evidence to reject H_0 .

Part 3. Hypothesis Testing for Mean (One-tailed test with $H_1: \mu < \mu_0$)

Graphical illustration of significance level vs. p-value for a one-tailed test (left-tailed):

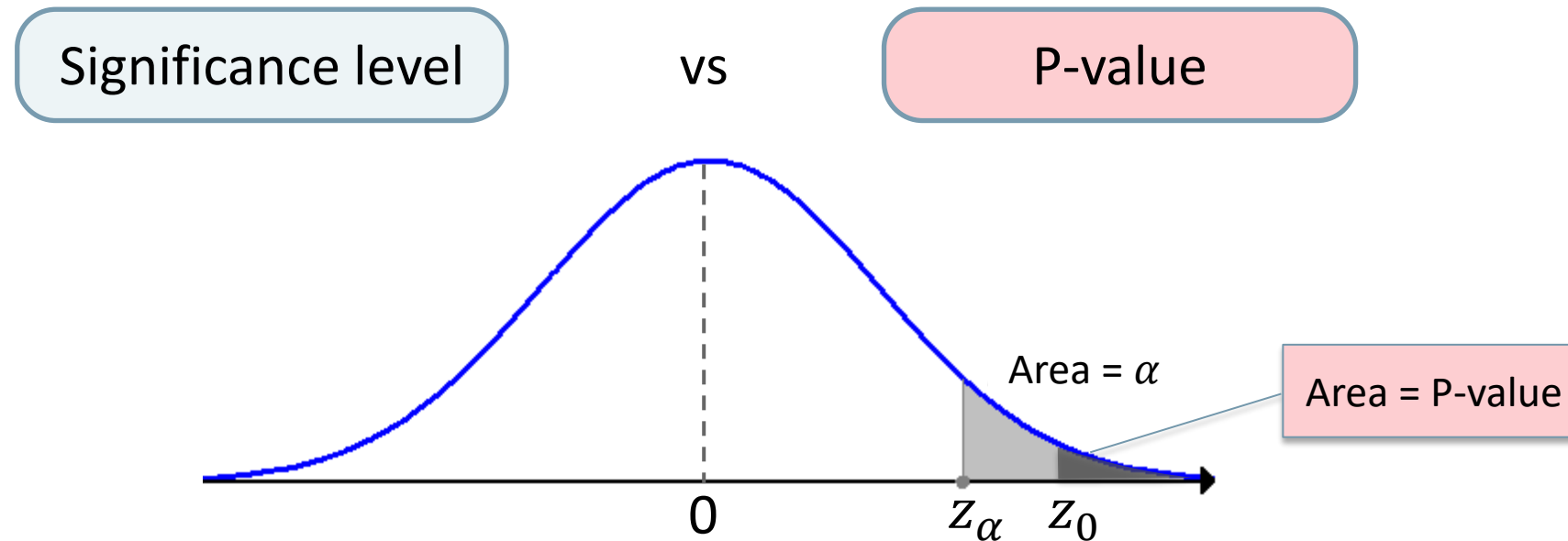


When the p-value $\leq \alpha$, the observed test statistic is sufficiently far from 0, which provides sufficient evidence to reject the null hypothesis H_0 .

Conversely, when the p-value $> \alpha$, there is insufficient evidence to reject H_0 .

Part 3. Hypothesis Testing for Mean (One-tailed test with $H_1: \mu > \mu_0$)

Graphical illustration of significance level vs. p-value for a one-tailed test (right-tailed):



When the p-value $\leq \alpha$, the observed test statistic is sufficiently far from 0, which provides sufficient evidence to reject the null hypothesis H_0 .

Conversely, when the p-value $> \alpha$, there is insufficient evidence to reject H_0 .

Part 3. Hypothesis Testing for Mean

HYPOTHESIS TESTING FOR MEAN

Step 1: State the null and alternative hypotheses.

Step 2: Calculate the test statistic.

Step 3: Calculate the p-value.

Step 4: State the conclusion by comparing p-value with the significance level:

There is sufficient/insufficient* evidence to reject the claim that [claim statement].

**sufficient if $p\text{-value} \leq \alpha$, insufficient if $p\text{-value} > \alpha$.*

Part 3. Hypothesis Testing for Mean

Case 1: Population Standard Deviation σ is known

When σ is known, we use the standard normal distribution (Z-distribution).

To understand why the Z-distribution can be used, we first consider the distribution of the sample mean $\bar{X}$:

THEOREM 3.1

If the population is normally distributed, the sample mean $\bar{X}$ is normally distributed with:

$$\text{Mean} = \mu \quad \text{and} \quad \text{Standard deviation} = \frac{\sigma}{\sqrt{n}} \quad \text{known}$$

It follows that $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ has a standard normal distribution with mean 0 and standard deviation 1.
known

➔ Therefore, when the population is normally distributed **and** σ is known, the Z-distribution can be used for hypothesis testing.

Part 3. Hypothesis Testing for Mean

But what if the population distribution is unknown (i.e. *whether normally distributed or not*)?

→ For a sufficiently large sample, the **Central Limit Theorem (CLT)** allows us to proceed even when the population distribution is unknown.

CENTRAL LIMIT THEOREM (CLT) - THEOREM 3.2

For a sufficiently large sample size ($n \geq 30$), the sample mean $\bar{X}$ is approximately normally distributed with: Mean = μ and Standard deviation = $\frac{\sigma}{\sqrt{n}}$

Therefore, if the pop. distribution is unknown, **but** the sample size is sufficiently large ($n \geq 30$), **and** the pop. standard deviation σ is known, then we can **calculate** test statistic using:

$$z_0 = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

Part 3. Hypothesis Testing for Mean

EXAMPLE 3.1

A hospital uses large amounts of packaged doses of a particular medication whose individual dosage is claimed to be 100 cm^3 by the manufacturer.

The manufacturer assumes the population standard deviation to be 5 cm^3 . A random sample of 49 doses from a recent shipment gave a mean dosage of 102 cm^3 .

Given a 1% level of significance, test the manufacturer's claim regarding drug dosage.

Hint: For this qns, let's conduct a two-tailed test!

Use this Excel Function for two-tailed z-test
 $=2*(1-\text{NORM.S.DIST}(\text{ABS}(z_0), \text{TRUE}))$

Step 1: $H_0: \mu = 100$
 $H_1: \mu \neq 100$

Step 2: $z_0 = \frac{102-100}{5/\sqrt{49}} = 2.8$

Step 3: P-value = 0.00511
 $\therefore \text{P-value} \leq 0.01$

Step 4: At a 1% level of significance, there is sufficient evidence to reject the claim that the medication individual dosage has a mean of 100 cm^3 .

Part 3. Hypothesis Testing for Mean

Case 2: Population Standard Deviation σ unknown

When σ is unknown, we use the sample standard deviation s and the t -distribution.

To understand why the t -distribution can be used, we first consider the distribution of the test statistic:

THEOREM 3.3

If the population is normally distributed, then $T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$ has a t -distribution with $(n - 1)$ degrees of freedom.

Hence, we **calculate** the test statistic using: $t_0 = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

→ Therefore, when the population is normally distributed and σ is unknown, the t -distribution can be used for hypothesis testing.

Part 3. Hypothesis Testing for Mean

But what if the population distribution is unknown (i.e. *whether normally distributed or not*), in addition to σ unknown?

Recall from Slide 29: For a sufficiently large sample ($n \geq 30$), the Central Limit Theorem (CLT) allows us to proceed even when the pop. distribution is unknown.

Recall from Slide 31: When σ is unknown, we instead use the sample standard deviation s (to estimate σ).

THEOREM 3.4

For a sufficiently large sample size ($n \geq 30$), when the pop. standard deviation σ is unknown, $T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$ is approximately t -distributed with $(n - 1)$ degrees of freedom.

Hence, we can still **calculate** the test statistic using:

$$t_0 = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Part 3. Hypothesis Testing for Mean

EXAMPLE 3.2

A manufacturer claims that the mean talk-time of its phone batteries is 80 hours fully charged.

A consumer watchdog believes the mean talk-time is less than 80 hours, because when they tested a random samples of 51 batteries from the manufacturer, it gave a sample mean of $\bar{x} = 78.83$ and sample standard deviation of $s = 9.04$.

Test the claim of the manufacturer at 5% significance level.

Hint: For this qns, let's conduct a left-tailed test!

Use this Excel Function for left-tailed t-test
 $=T.DIST(t_0, n - 1, TRUE)$

Step 1: $H_0: \mu = 80$
 $H_1: \mu < 80$ ($\leftarrow$ Why?)

Step 2: $t_0 = \frac{78.83 - 80}{9.04/\sqrt{51}} = -0.9243$

Step 3: P-value = 0.1798
 $\therefore$ P-value > 0.05

Step 4: At a 5% level of significance, there is insufficient evidence to reject the manufacturer's claim that the mean talk-time of its phone batteries is 80 hours when fully charged.

Part 4. Confidence Interval

CONFIDENCE INTERVAL FOR MEAN

Suppose one the following conditions is met: the sample size $n \geq 30$, or the population is normally distributed.

Given a significance level α , the corresponding confidence level is: $100(1 - \alpha)\%$,
and the **confidence interval** for μ is: $\bar{x} - E_\alpha < \mu < \bar{x} + E_\alpha$
where E_α is the *margin of error* associated with α .

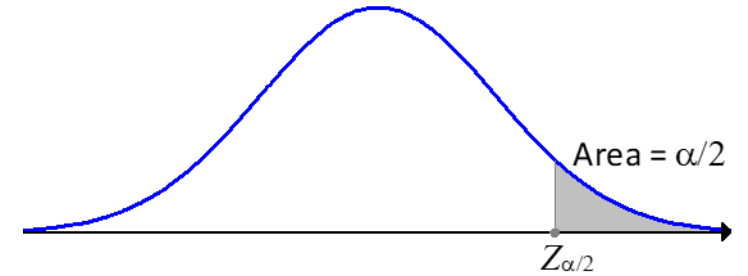
For example, given $\alpha = 5\%$, the corresponding 95% confidence interval for the population mean μ lies
between $\bar{x} - E_{0.05}$ and $\bar{x} + E_{0.05}$,
i.e. $\bar{x} - E_{0.05} < \mu < \bar{x} + E_{0.05}$

The $100(1 - \alpha)\%$ confidence interval for μ is written as $\bar{x} \pm E_\alpha$.

Part 4. Confidence Interval

The margin of error E_α is computed as follows:

If σ is known, then $E_\alpha = (z_{\alpha/2}) \frac{\sigma}{\sqrt{n}}$ with $P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$



If σ is unknown, then $E_\alpha = (t_{\alpha/2}) \frac{s}{\sqrt{n}}$ with $P(T > t_{\alpha/2}) = \frac{\alpha}{2}$

Good news! → In practice, we can use a computer to find the value of E_α .

For instance, in Microsoft Excel, we use the statistical function

=CONFIDENCE.NORM (when σ is known) **or** **=CONFIDENCE.T** (when σ is unknown).

Part 4. Confidence Interval

EXAMPLE 4.1

A random sample of 100 observations with a sample mean 22.5 is taken from a population.

Find the 90% confidence interval for μ if

- (a) The population standard deviation is 0.5,
- (b) The sample standard deviation is 0.75.

	A	B
1	n =	100
2	α =	0.1
3	σ =	0.5
4	s =	0.75
5	E =	0.082243 =CONFIDENCE.NORM(B2,B3,B1)
6	E =	0.124529 =CONFIDENCE.T(B2,B4,B1)

$$1 - \alpha = 0.90$$

$$\Rightarrow \alpha = 0.1$$

(a) $E_{0.1} = 0.082$

90% confidence interval for μ is

$$22.5 - 0.082 < \mu < 22.5 + 0.082$$

$$\Rightarrow 22.418 < \mu < 22.582$$

(b) $E_{0.1} = 0.125$

90% confidence interval for μ is

$$22.5 - 0.125 < \mu < 22.5 + 0.125$$

$$\Rightarrow 22.375 < \mu < 22.625$$

Summary

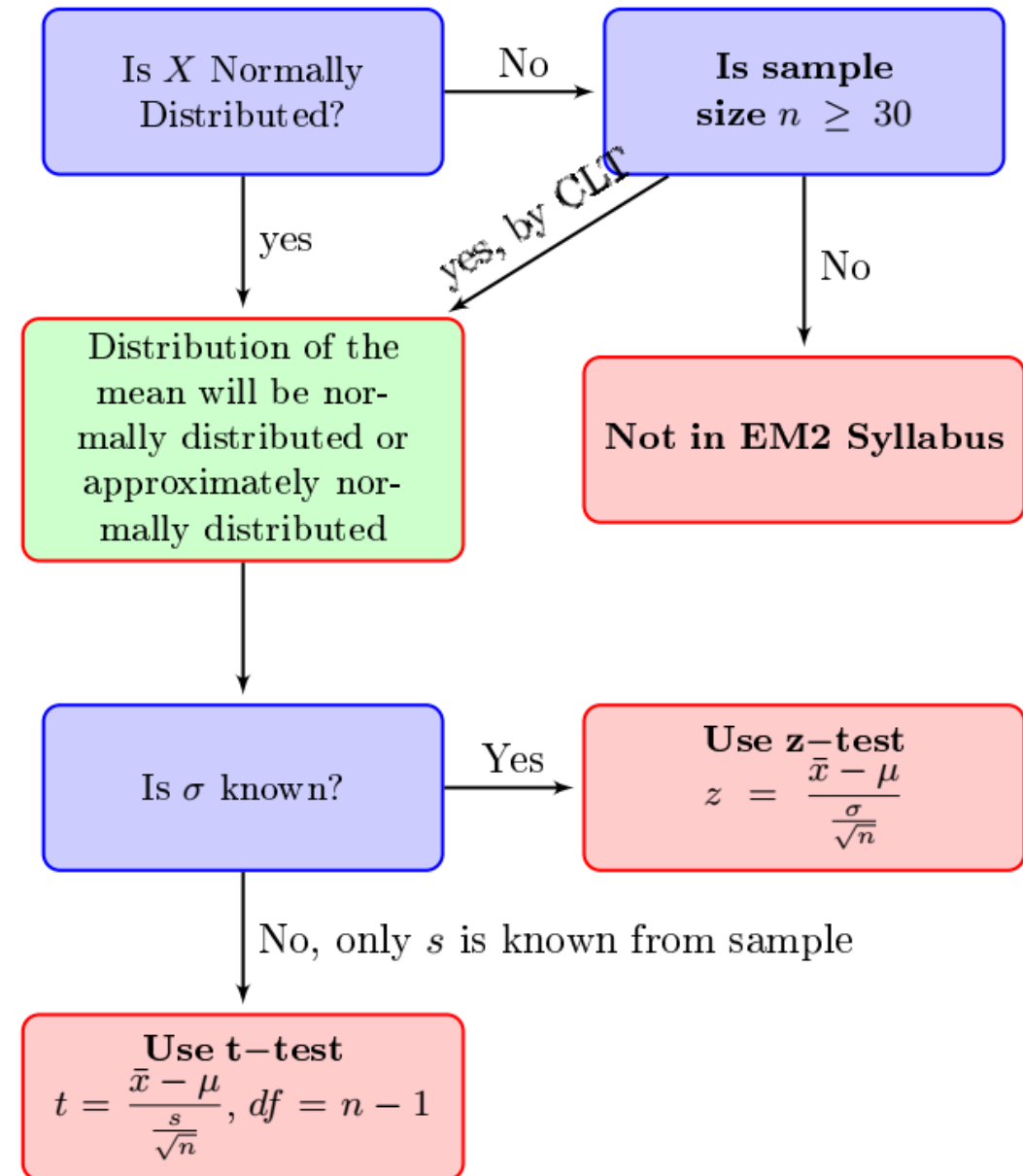
Condition	σ	Confidence Interval for Mean	Hypothesis Testing for Mean	
		$\bar{x} - E_{\alpha} < \mu < \bar{x} + E_{\alpha}$	Test Statistic	P-value * The formulas below are for two-tailed test, adjust accordingly when performing one-tail test (L or R)
Population is Normally distributed OR Sample size $n \geq 30$	Known	$E_{\alpha} = (z_{\alpha/2}) \frac{\sigma}{\sqrt{n}}$ =CONFIDENCE.NORM(α, σ, n)	$z_0 = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$	$2P(z > z_0)$ =2*(1-NORM.S.DIST($ z_0 $,TRUE))
	Unknown	$E_{\alpha} = (t_{\alpha/2}) \frac{s}{\sqrt{n}}$ =CONFIDENCE.T(α, s, n)	$t_0 = \frac{\bar{x} - \mu}{s / \sqrt{n}}$	$2P(t > t_0)$ =2*(T.DIST.RT($ t_0 , n - 1$))

Flowchart for Hypothesis Test

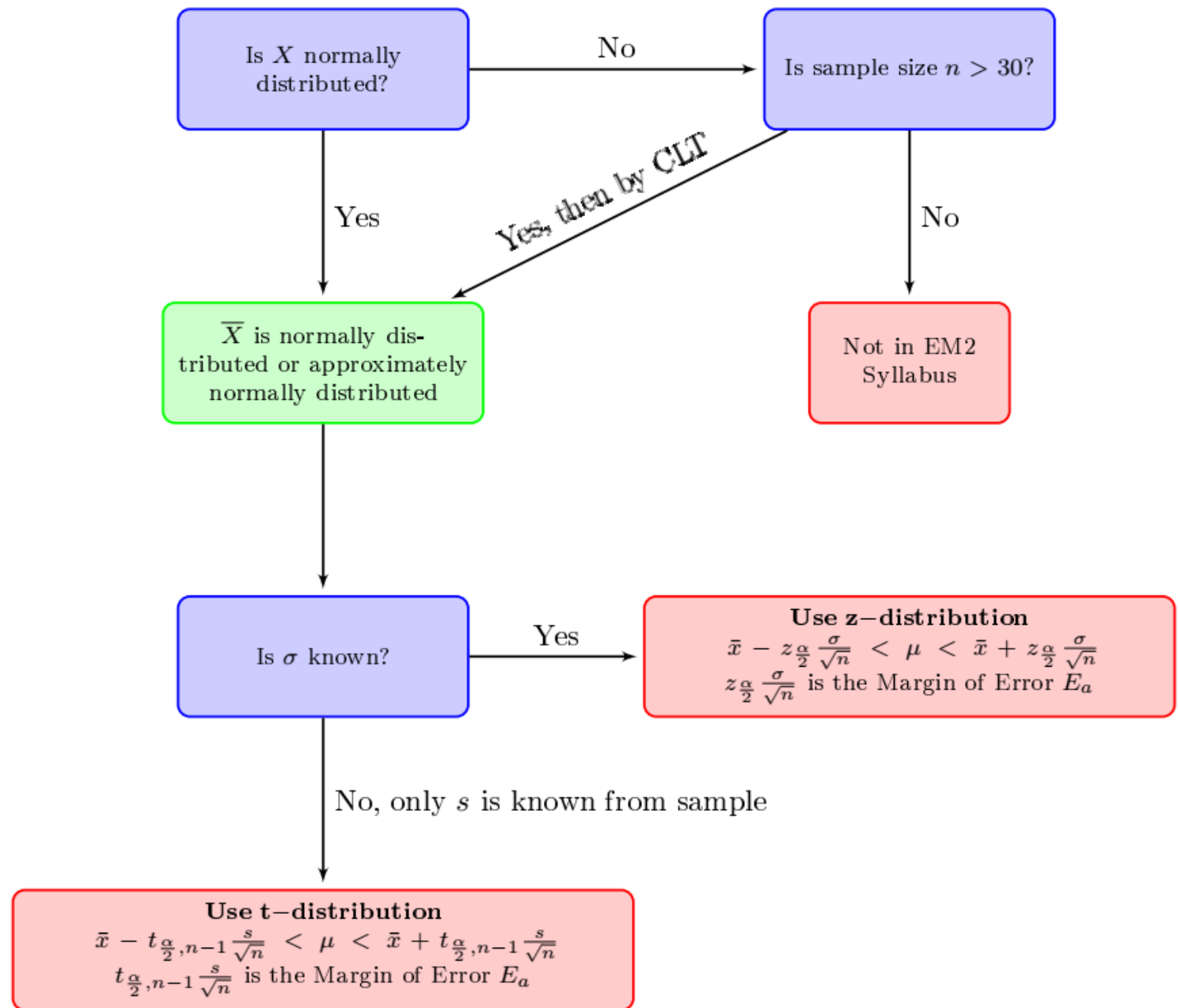
- Step 1:** State the null and alternative hypotheses.
- Step 2:** Calculate the test statistic.
- Step 3:** Calculate the p-value.
- Step 4:** State the conclusion by comparing p-value with the significance level:

There is sufficient/insufficient* evidence to reject the claim that [claim statement].

**sufficient if $p\text{-value} \leq \alpha$,
insufficient if $p\text{-value} > \alpha$*



Flowchart for Confidence Interval



Reference

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